Disjoint Sets: Naive Implementations

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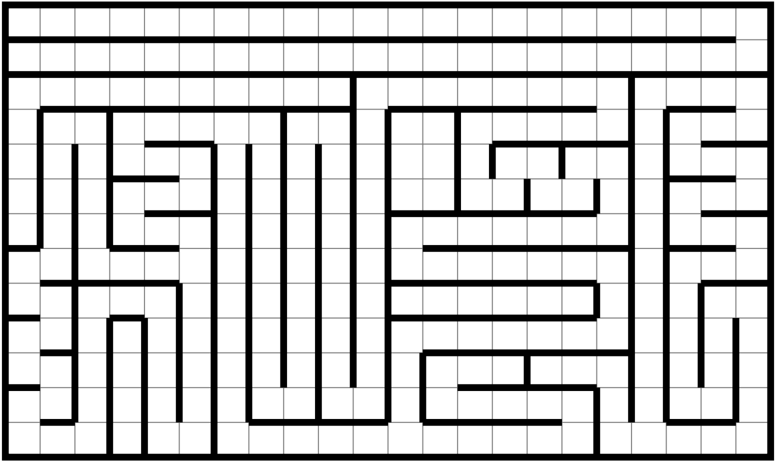
Data Structures
Data Structures and Algorithms

Outline

Maze: Is B Reachable from A?



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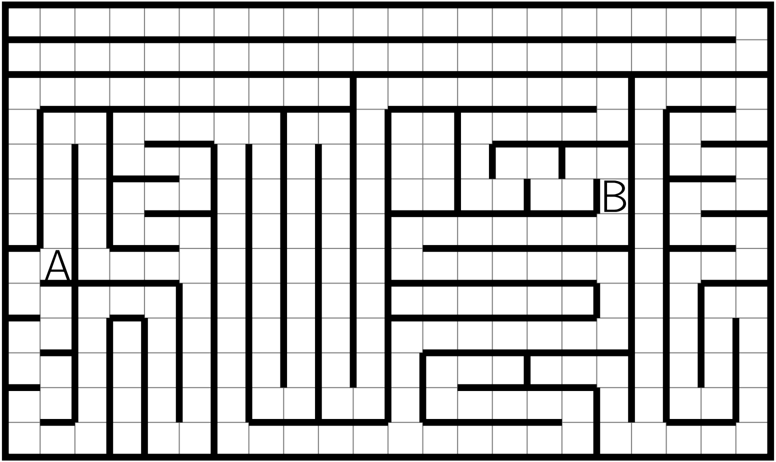

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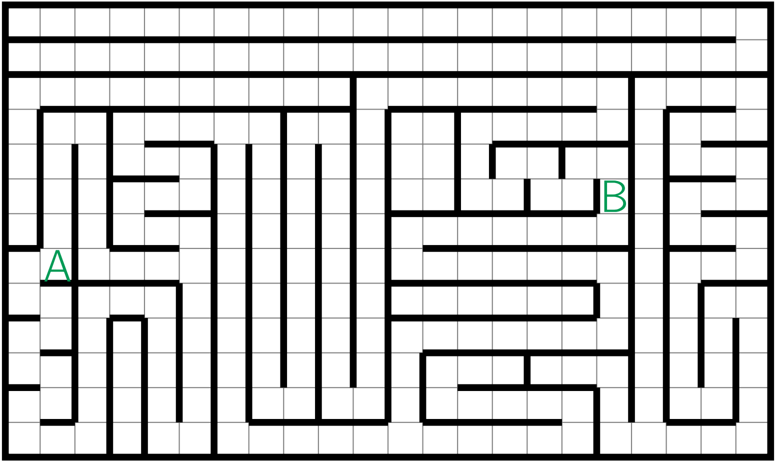

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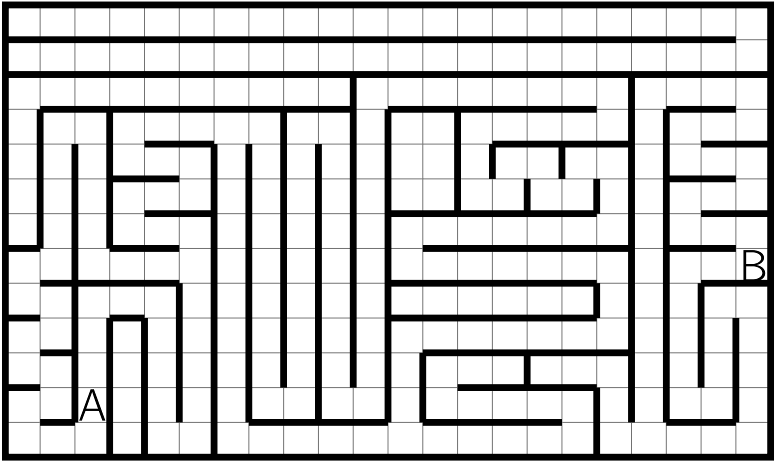

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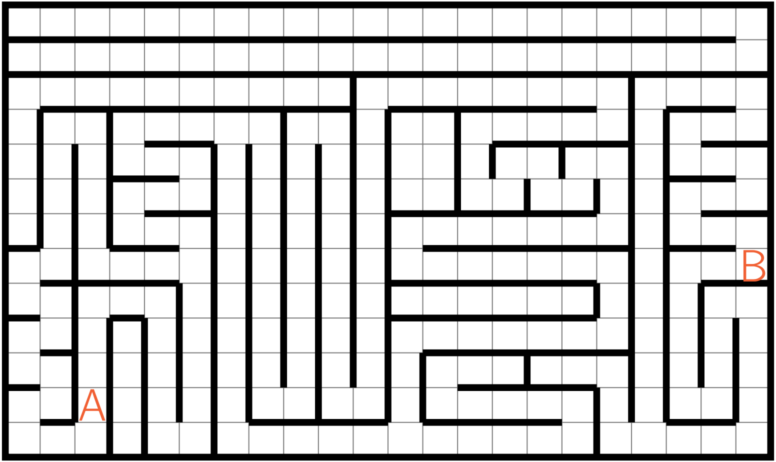

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  - otherwise, $\texttt{Find}(x) \neq \texttt{Find}(y)$
- $\texttt{Union}(x, y)$ merges two sets containing x and y

Preprocess(*maze*)

```
for each cell c in maze:
  MakeSet(c)
for each cell c in maze:
  for each neighbor n of c:
    Union(c, n)
```

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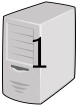
IsReachable(A, B)

```
return Find(A) = Find(B)
```

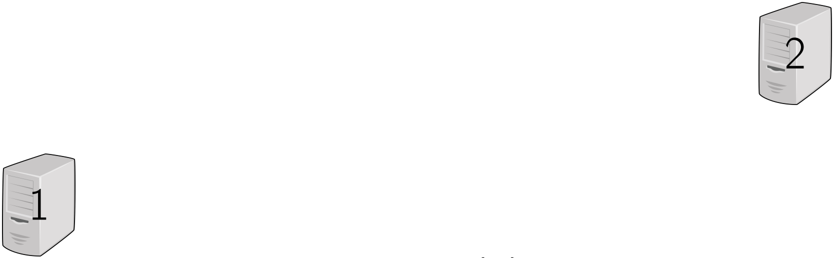
Building a Network

Building a Network



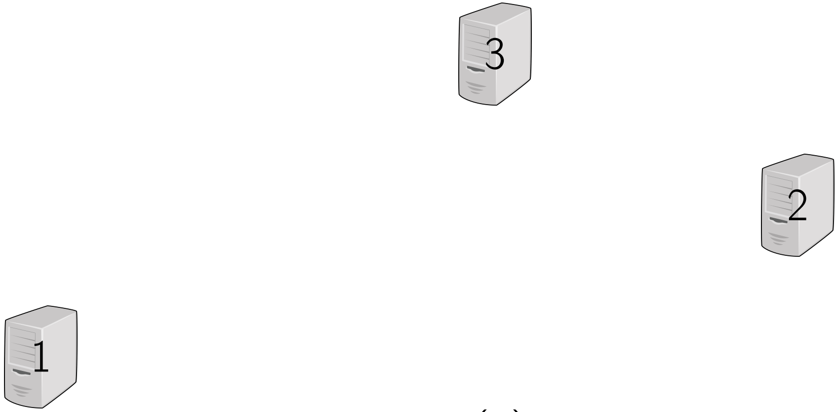
`MakeSet(1)`

Building a Network



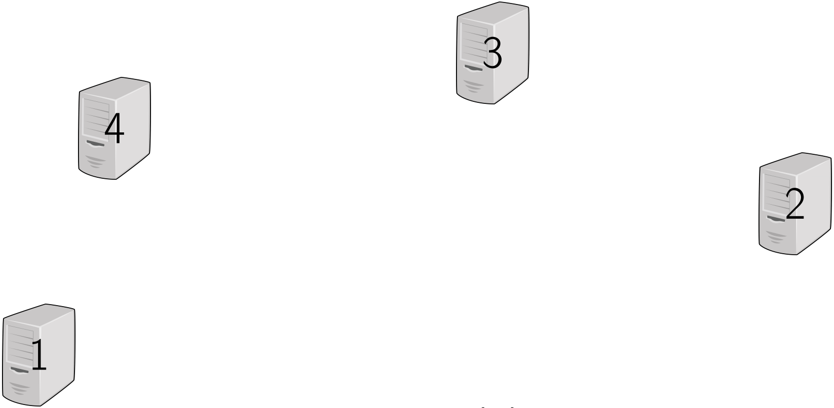
`MakeSet(2)`

Building a Network



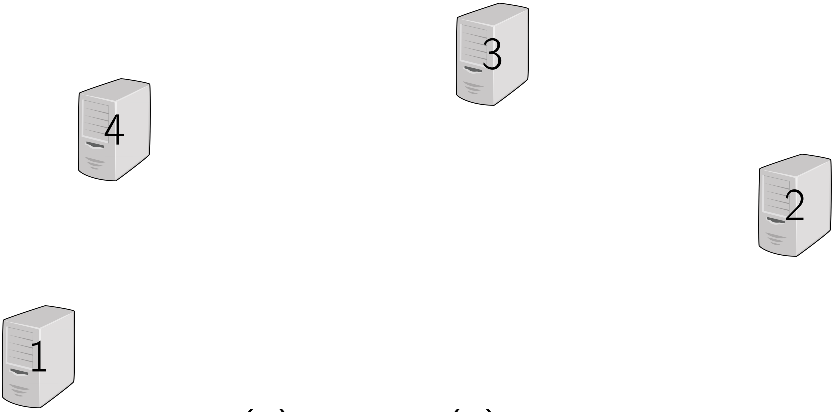
`MakeSet(3)`

Building a Network



`MakeSet(4)`

Building a Network



$$\texttt{Find}(1) = \texttt{Find}(2) \to \texttt{False}$$

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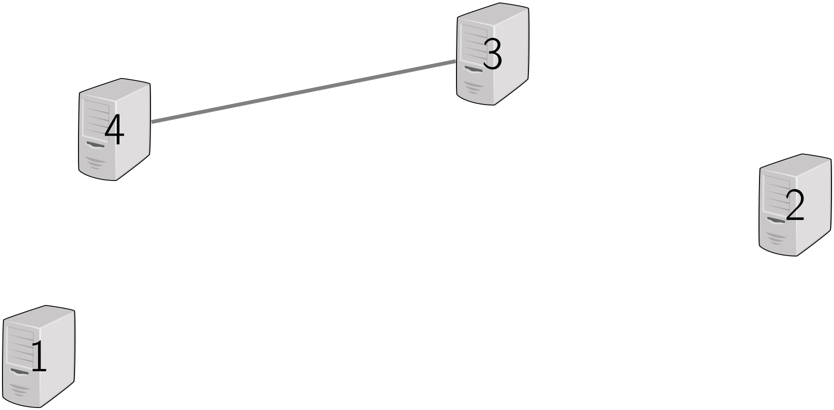

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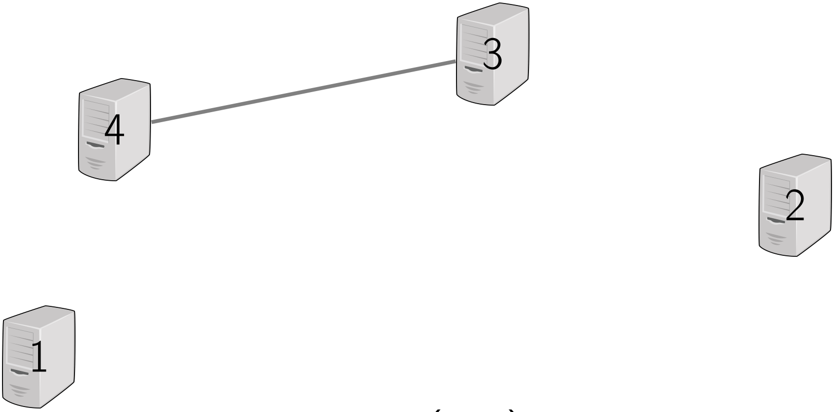
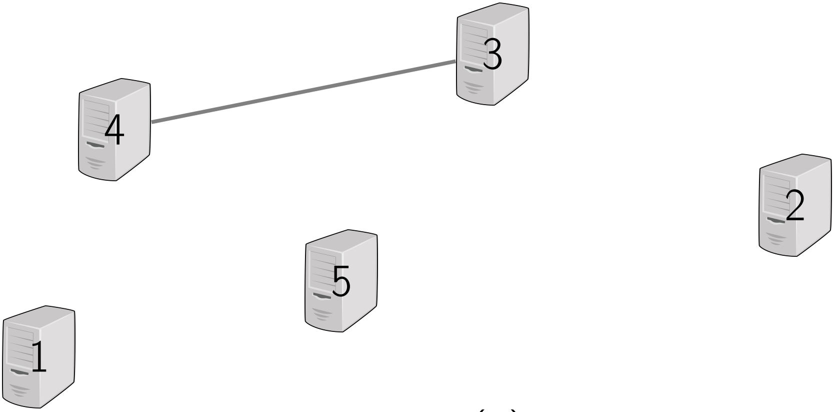


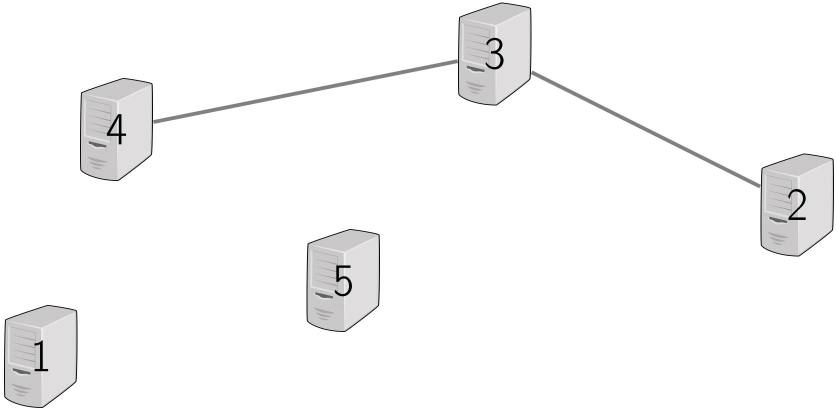
`Union(3,4)`

Building a Network



`MakeSet(5)`

Building a Network



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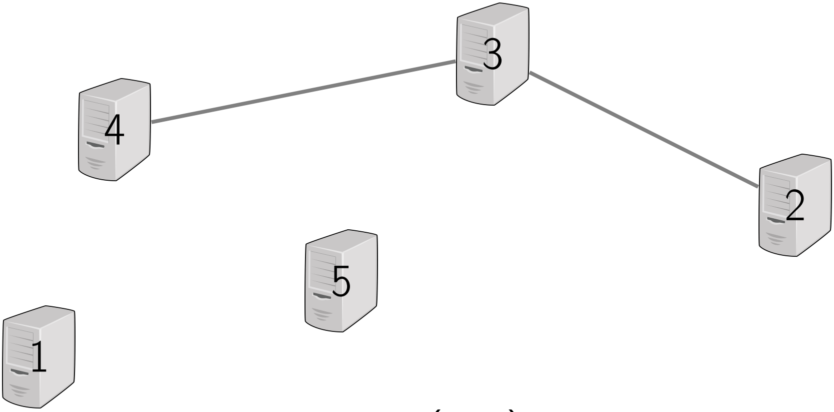
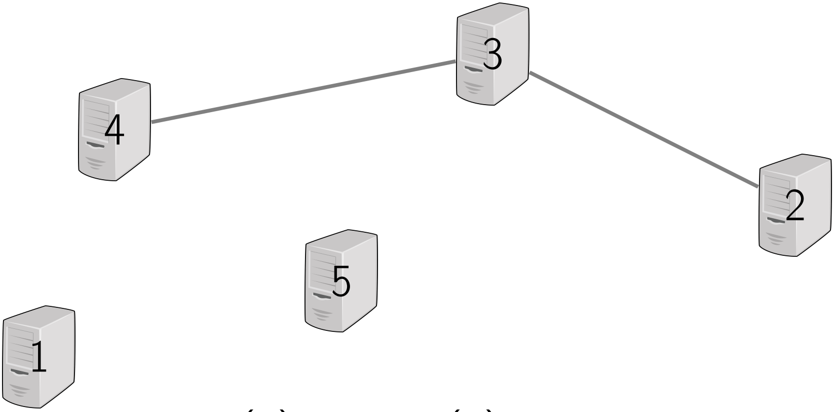


`Union(3,2)`

Building a Network



$$\texttt{Find}(1) = \texttt{Find}(2) \to \texttt{False}$$

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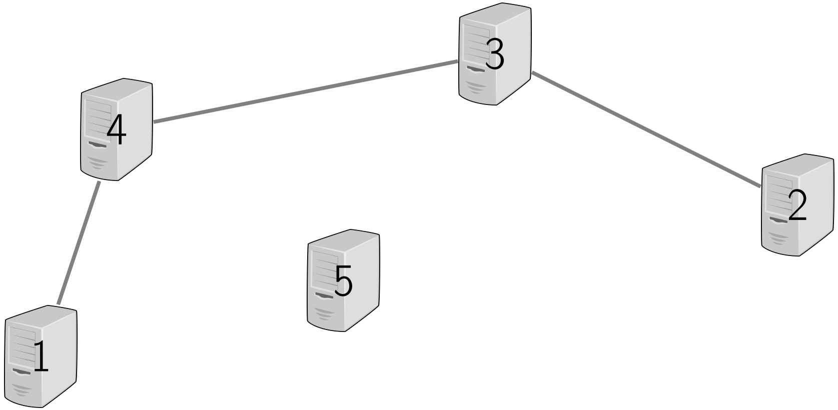

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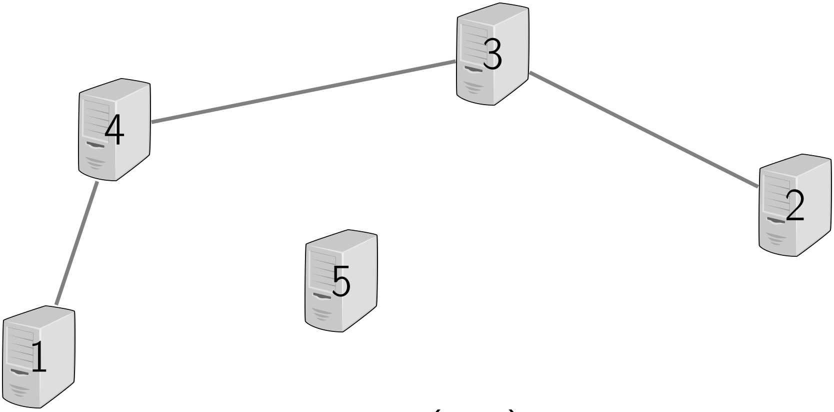
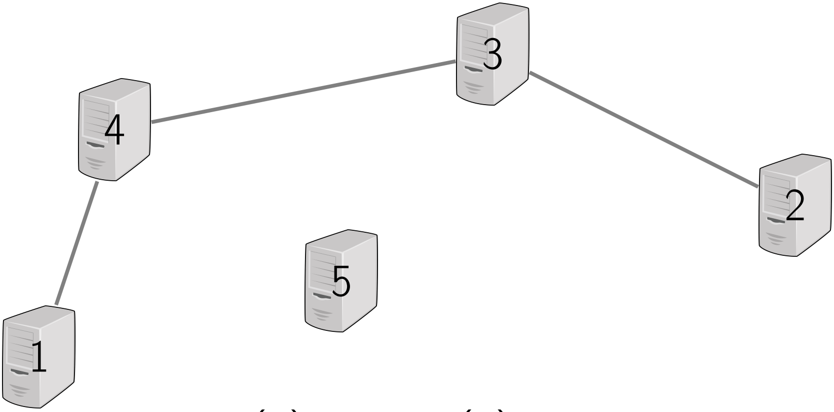


`Union(1,4)`

Building a Network



$\texttt{Find}(1) = \texttt{Find}(2) \rightarrow \texttt{True}$

Outline

1. Overview
2. Naive Implementations

For simplicity, we assume that our n objects are just integers $1, 2, \ldots, n$.

Using the Smallest Element as ID

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- Use array `smallest`$[1 \dots n]$: `smallest`$[i]$ stores the smallest element in the set i belongs to

Example

$$\{9, 3, 2, 4, 7\} \quad \{5\} \quad \{6, 1, 8\}$$

	1	2	3	4	5	6	7	8	9
`smallest`	1	2	2	2	5	1	2	1	2

MakeSet(i)

```
smallest[i] ← i
```

Find(i)

```
return smallest[i]
```

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Running time: $O(1)$

$\texttt{Union}(i, j)$

$i\_id \leftarrow \texttt{Find}(i)$
$j\_id \leftarrow \texttt{Find}(j)$
$\texttt{if } i\_id = j\_id:$
  $\texttt{return}$
$m \leftarrow \min(i\_id, j\_id)$
$\texttt{for } k \texttt{ from 1 to } n:$
  $\texttt{if smallest}[k] \texttt{ in } \{i\_id, j\_id\}:$
    $\texttt{smallest}[k] \leftarrow m$

$\texttt{Union}(i, j)$

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i_id ← Find(i)
j_id ← Find(j)
if i_id = j_id:
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m ← min(i_id, j_id)
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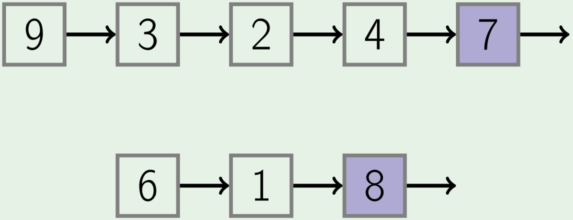
- What basic data structure allows for efficient merging?
- Linked list!
- Idea: represent a set as a linked list, use the list tail as ID of the set

Example: merging two lists



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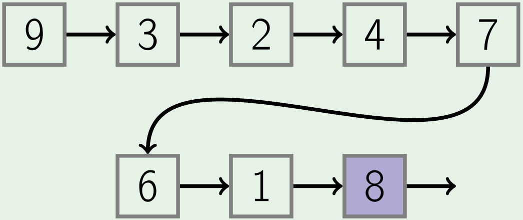

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  - Running time of `Find` is $O(n)$ as we need to traverse the list to find its tail
  - `Union`(x, y) works in time $O(1)$ only if we can get the tail of the list of x and the head of the list of y in constant time!

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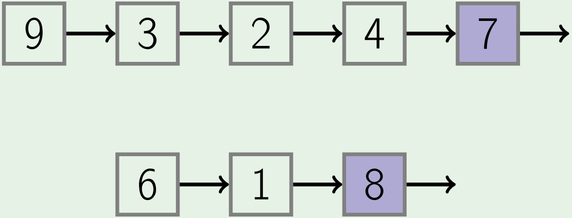

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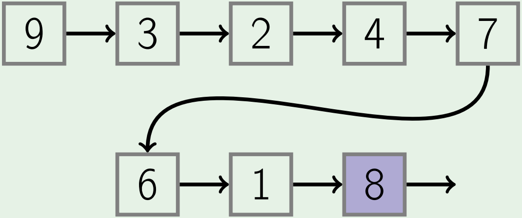

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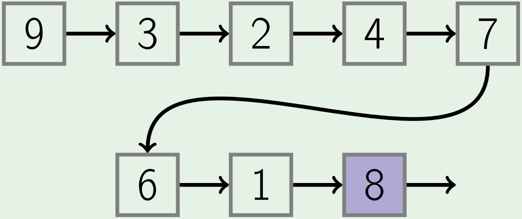

`Find`(9) goes through all elements

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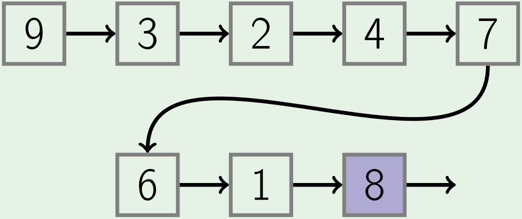

can we merge in a different way?

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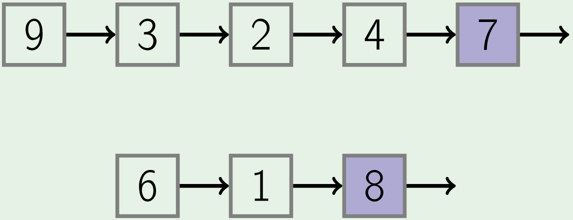

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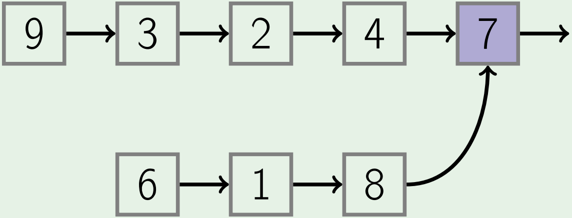

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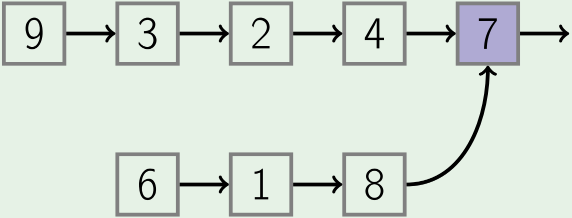

instead of a list we get a tree

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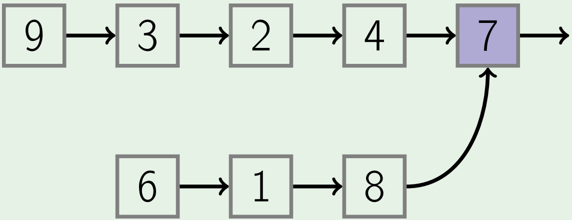

we'll see that representing sets as trees gives a very efficient implementation: nearly constant amortized time for all operations