Introduction:

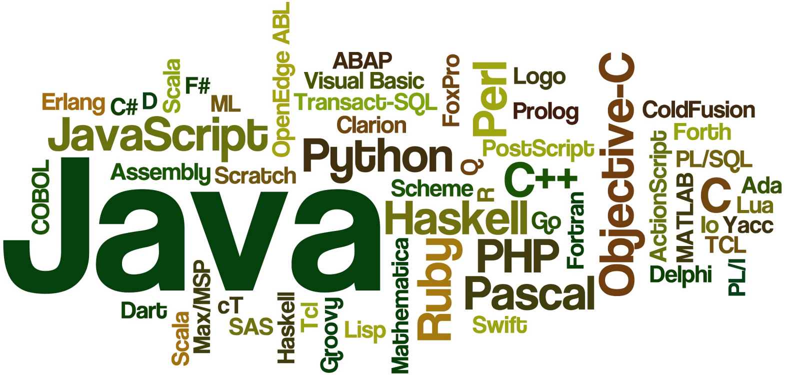
Michael Levin

Hash Tables

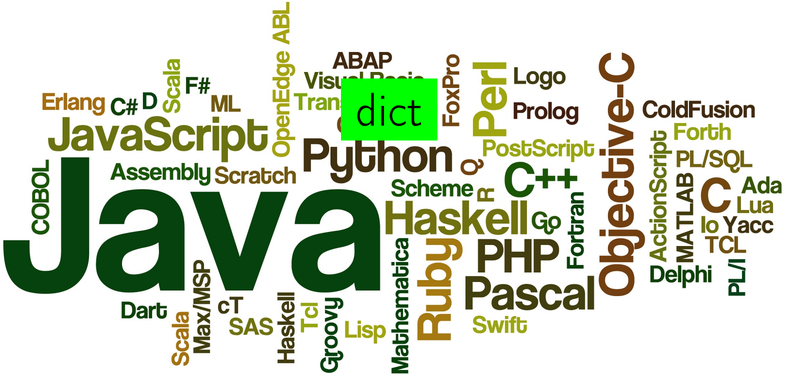
Data Structures and Algorithms
Algorithmic Toolbox

Outline

Programming Languages



Programming Languages



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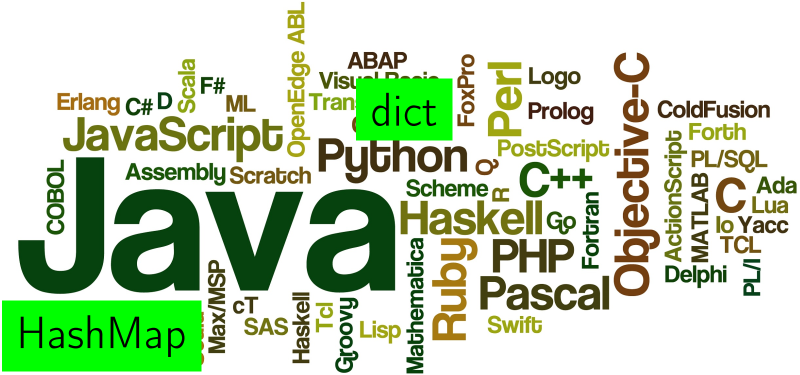

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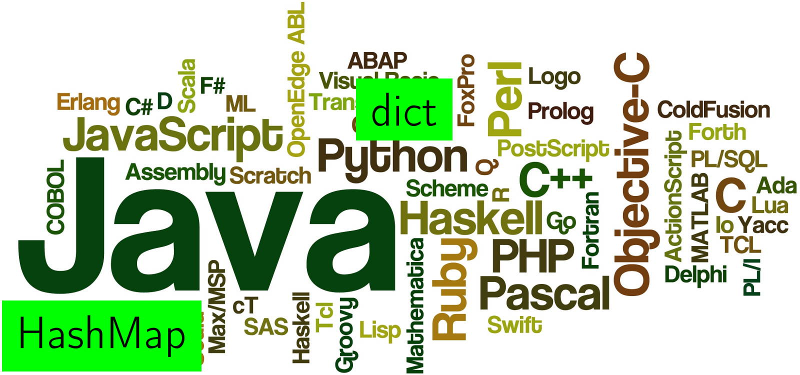
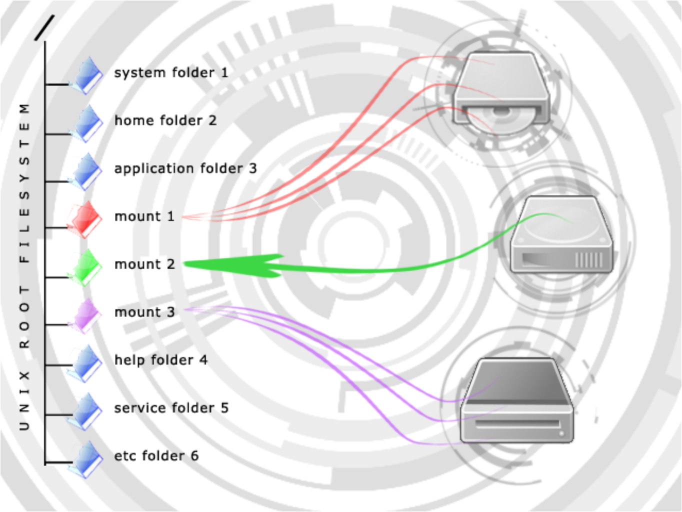
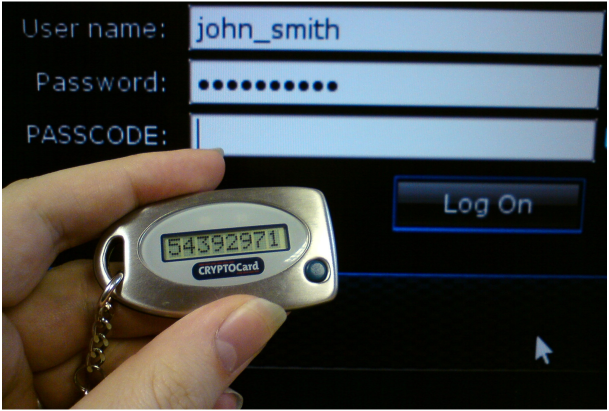


for, if, while, int

File Systems



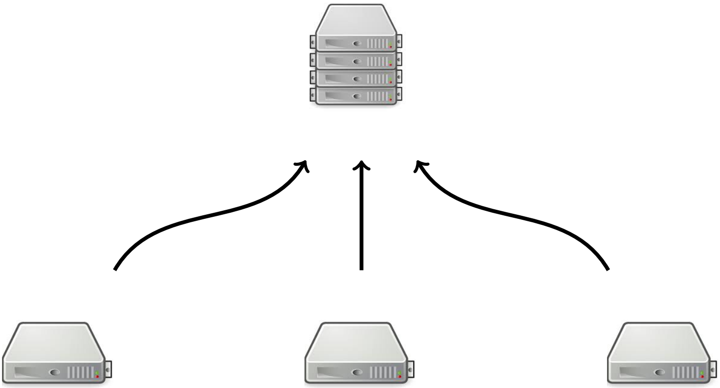
Password Verification



Storage Optimization

Outline

Web Service



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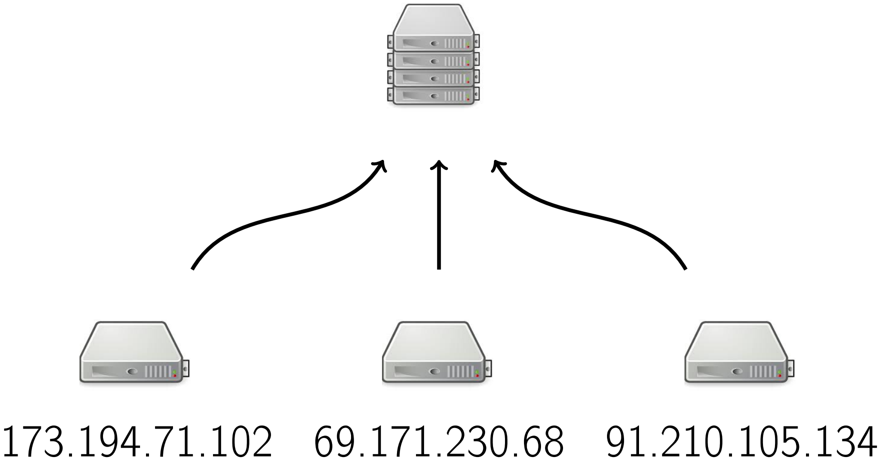
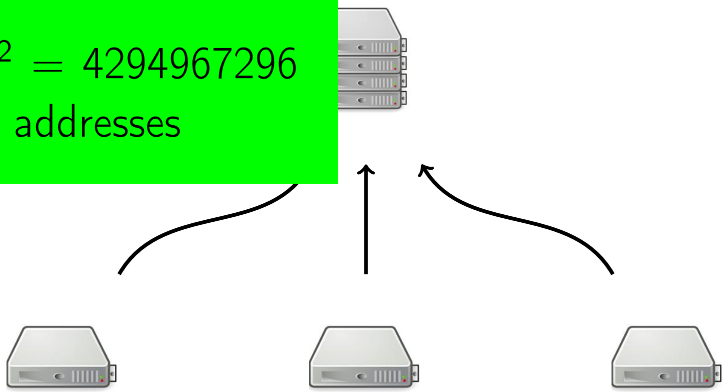


Web Service

$2^{32} = 4294967296$

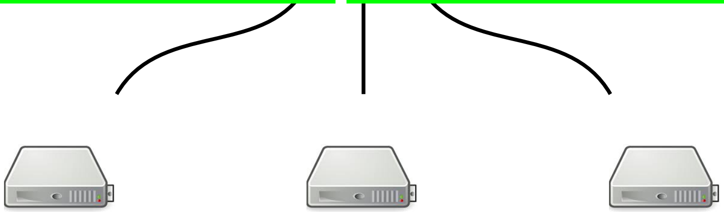
IP addresses



Web Service

$2^{32} = 4294967296$ IP addresses

2^{128} IPv6 addresses — number with 39 digits!



Access Log

Date	Time	IP address
09 Dec 2015	00:45:13	173.194.71.102
09 Dec 2015	00:45:15	69.171.230.68
...	...	...
...	...	...
09 Dec 2015	01:45:13	91.210.105.134

IP Access List

Analyse the access log and quickly answer queries: did anybody access the service from this *IP* during the last hour? How many times? How many *IP*s were used to access the service during the last hour?

Log Processing

- 1h of logs can contain millions of lines

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- C is some data structure to store the mapping from IPs to counters
- We will learn later how to implement C

Log Processing

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Log Processing

Now

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Increment counter

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1 hour ago

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Increment counter

Log Processing

Decrement counter

1 hour ago

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Now

Increment counter

Main Loop

log - array of log lines $(time, IP)$

C - mapping from IPs to counters

i - first unprocessed log line

j - first line in current 1h window

$i \leftarrow 0$

$j \leftarrow 0$

$C \leftarrow \emptyset$

Each second

    UpdateAccessList(log, i, j, C)

$\texttt{UpdateAccessList}(\mathit{log}, i, j, C)$

$\texttt{while}\ \mathit{log}[i].\mathit{time} \leq \mathit{Now}()$:
  $C[\mathit{log}[i].\mathit{IP}] \leftarrow C[\mathit{log}[i].\mathit{IP}] + 1$
  $i \leftarrow i + 1$
$\texttt{while}\ \mathit{log}[j].\mathit{time} \leq \mathit{Now}() - 3600$:
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$\texttt{AccessedLastHour}(IP, C)$

$\texttt{return}\ C[IP] > 0$

Coming Next

How to implement the mapping C?

Outline

1. Applications of Hashing
2. IP Addresses
3. Direct Addressing
4. List-based Mapping
5. Hash Functions
6. Chaining
7. Hash Tables

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- Need a data structure for C
- There are 2^{32} different IP(v4) addresses
- Convert IP to 32-bit integer
- Create an integer array A of size 2^{32}
- Use $A[\texttt{int}(IP)]$ as $C[IP]$

int(IP)

An IPv4 address (dotted-decimal notation)

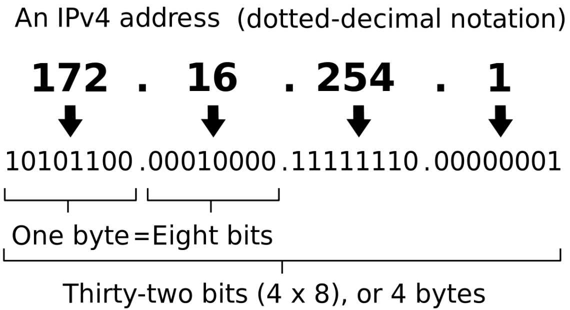
172 . 16 . 254 . 1

10101100.00010000.11111110.00000001

One byte = Eight bits

Thirty-two bits (4 x 8), or 4 bytes

int(IP)



- $\mathtt{int}(0.0.0.1) = 1$

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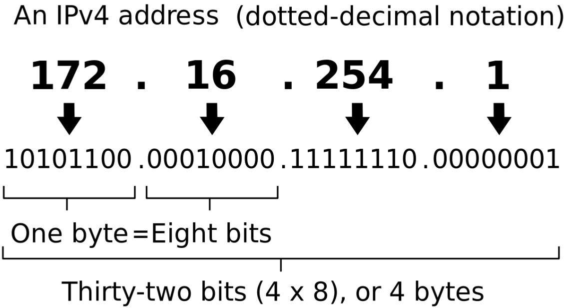

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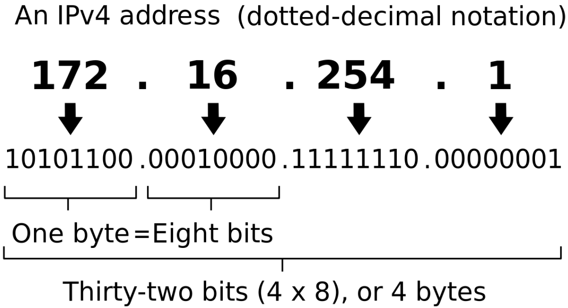

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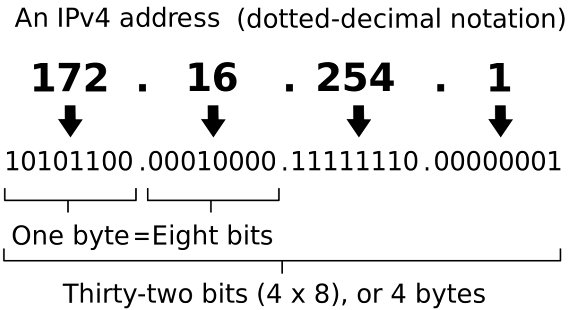

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$\texttt{return}\ \ IP[1] \cdot 2^{24} + IP[2] \cdot 2^{16} + IP[3] \cdot 2^{8} + IP[4]$

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- In general: $O(N)$ memory, $N = |S|$

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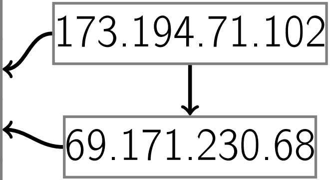

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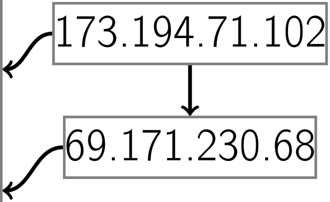

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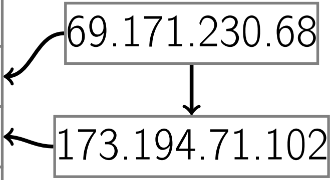

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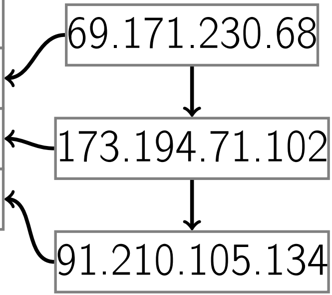

```
UpdateAccessList(log, i, L)

while log[i].time ≤ Now():
    log_line ← L.FindByIP(log[i].IP)
    if log_line ≠ NULL:
        L.Erase(log_line)
    L.Append(log[i])
    i ← i + 1
while L.Top().time ≤ Now() − 3600:
    L.Pop()
```

$\texttt{AccessedLastHour}(IP, L)$

$\texttt{return } L.\texttt{FindByIP}(IP) \neq NULL$

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- I.e. numbers from 0 to 999
- Different codes for currently active IPs

Hash Function

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For any set of objects S and any integer $m > 0$, a function $h : S \to \{0, 1, \ldots, m-1\}$ is called a hash function.

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- Different values for different objects
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- Want small cardinality m
- Impossible to have all different values if number of objects $|S|$ is more than m

Collisions

Definition

When $h(o_1) = h(o_2)$ and $o_1 \neq o_2$, this is a collision.

Outline

Map

Store mapping from objects to other objects:

- Filename $\rightarrow$ location of the file on disk
- Student ID $\rightarrow$ student name
- Contact name $\rightarrow$ contact phone number

Map

Store mapping from objects to other objects:

- Filename $\to$ location of the file on disk
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- Contact name $\to$ contact phone number

Definition

Map from S to V is a data structure with methods $\texttt{HasKey}(O)$, $\texttt{Get}(O)$, $\texttt{Set}(O, v)$, where $O \in S, v \in V$.

Chaining

0
1
2
3
4
5
6
7

Chaining

$$h(173.194.71.102) = 4$$

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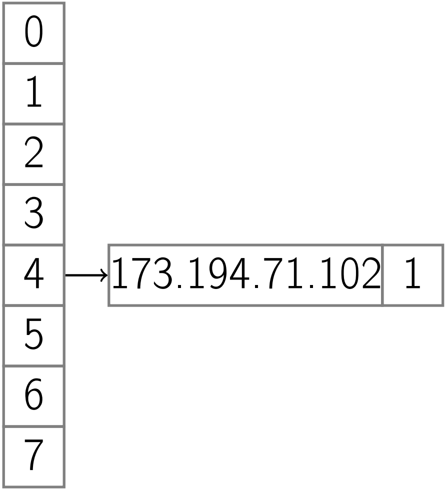
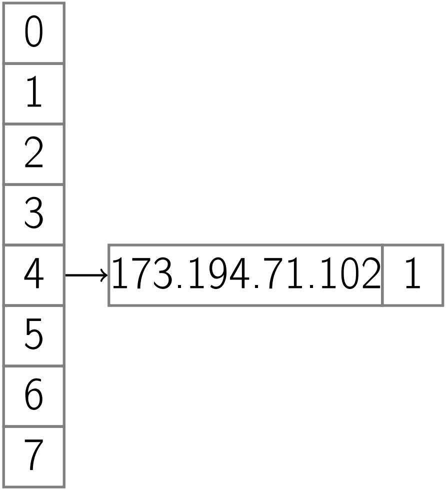


Chaining

$$h(173.194.71.102) = 4$$

$$h(69.171.230.68) = 1$$



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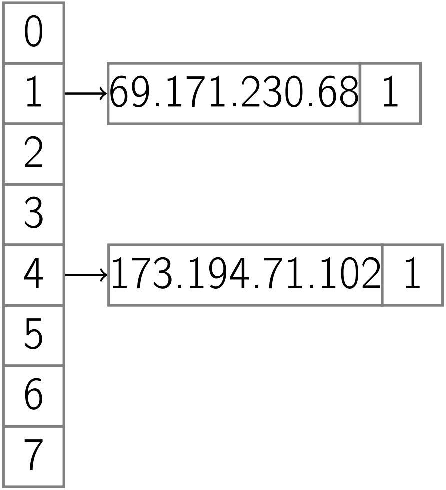

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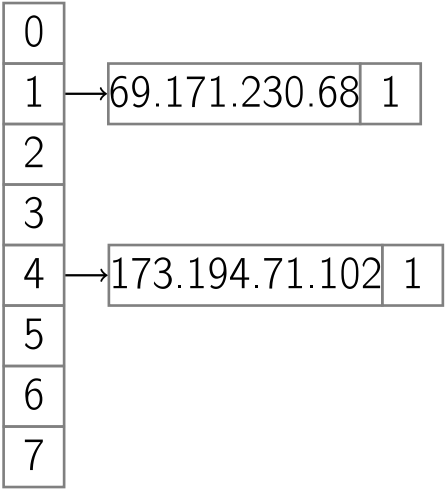

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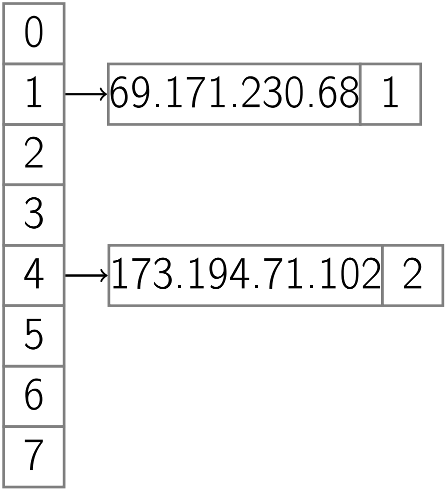

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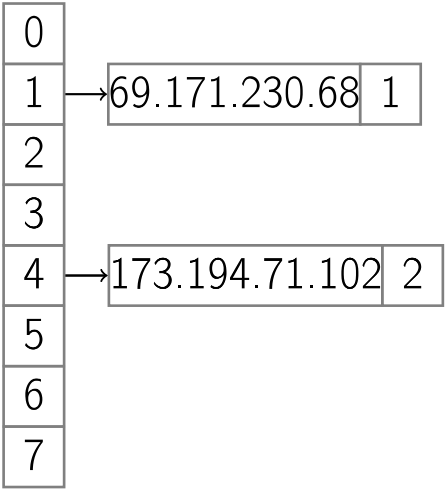

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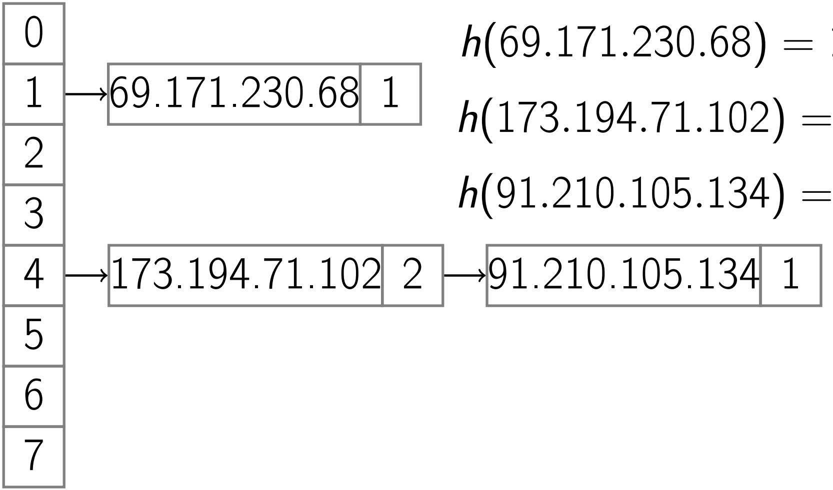

$h : S \to \{0, 1, \ldots, m-1\}$
$O, O' \in S$
$v, v' \in V$
$A \leftarrow$ array of m lists (chains) of pairs (O, v)

`HasKey`(O)

```
L ← A[h(O)]
for (O', v') in L:
  if O' == O:
    return true
return false
```

$\texttt{Get}(O)$

```
L ← A[h(O)]
for (O′, v′) in L:
    if O′ == O:
        return v′
return n/a
```

$\texttt{Set}(O, v)$

```
L ← A[h(O)]
for p in L:
    if p.O == O:
        p.v ← v
        return
L.Append(O, v)
```

Lemma

Let c be the length of the longest chain in A. Then the running time of `HasKey`, `Get`, `Set` is $\Theta(c + 1)$.

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- If $c = 0$, we still need $O(1)$ time $\square$

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- $\Theta(n)$ to store n pairs (O, v)
- $\Theta(m)$ to store array A of size m $\square$

Outline

Set

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- Keywords in a programming language

Implementing Set

Two ways to implement a set using chaining:

- Set is equivalent to map from S to $V = \{\mathit{true}, \mathit{false}\}$

Implementing Set

Two ways to implement a set using chaining:

- Set is equivalent to map from S to $V = \{true, false\}$
- Store just objects O instead of pairs (O, v) in chains

$h : S \to \{0, 1, \ldots, m-1\}$

$O, O' \in S$

$A \leftarrow$ array of m lists (chains) of objects O

Find(O)

```
L ← A[h(O)]
for O' in L:
  if O' == O:
    return true
return false
```

$\texttt{Add}(O)$

```
L ← A[h(O)]
for O' in L:
    if O' == O:
        return
L.Append(O)
```

Remove(O)

```
if not Find(O):
  return
L ← A[h(O)]
L.Erase(O)
```

Hash Table

Definition

An implementation of a set or a map using hashing is called a hash table.

Programming Languages

Set:

- `unordered_set` in C++
- `HashSet` in Java
- `set` in Python

Map:

- `unordered_map` in C++
- `HashMap` in Java
- `dict` in Python

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- Memory consumption is $O(n+m)$
- Operations work in time $O(c+1)$
- How to make both m and c small?