Hash Tables: Hash Functions

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Higher School of Economics

Data Structures
Data Structures and Algorithms

Outline

1. Good Hash Functions
2. Universal Family
3. Hashing Integers
4. Hashing Strings

Phone Book

Design a data structure to store your contacts: names of people along with their phone numbers. The data structure should be able to do the following quickly:

- Add and delete contacts,
- Lookup the phone number by name,
- Determine who is calling given their phone number.

- We need two Maps:
  (phone number $\rightarrow$ name) and
  (name $\rightarrow$ phone number)

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  (phone number $\rightarrow$ name) and
  (name $\rightarrow$ phone number)
- Implement these Maps as hash tables
- First, we will focus on the Map from phone numbers to names

Direct Addressing

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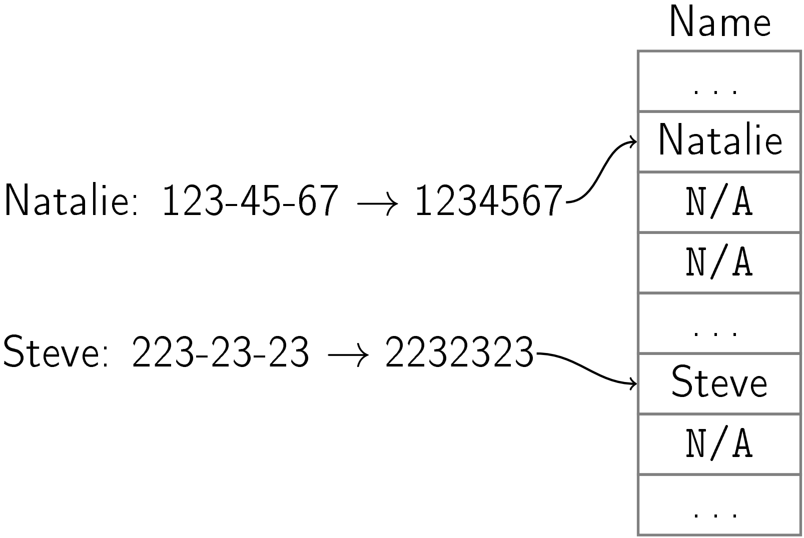
- Create array *Name* of size 10^L where L is the maximum allowed phone number length
- Store the name corresponding to phone number P in $Name[\texttt{int}(P)]$
- If no contact with phone number P, $Name[\texttt{int}(P)] = \texttt{N/A}$

Direct Addressing



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- Memory usage: $O(10^L)$, where L is the maximum length of a phone number
- Problematic with international numbers of length 12 and more: we will need 10^{12} bytes = 1TB to store one person's phone book — this won't fit in anyone's phone!

Chaining

- Select hash function h with cardinality m

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- Store chains in each cell of the array $Name$

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- Store chains in each cell of the array $Name$
- Chain $Name[h(\texttt{int}(P))]$ contains the name for phone number P

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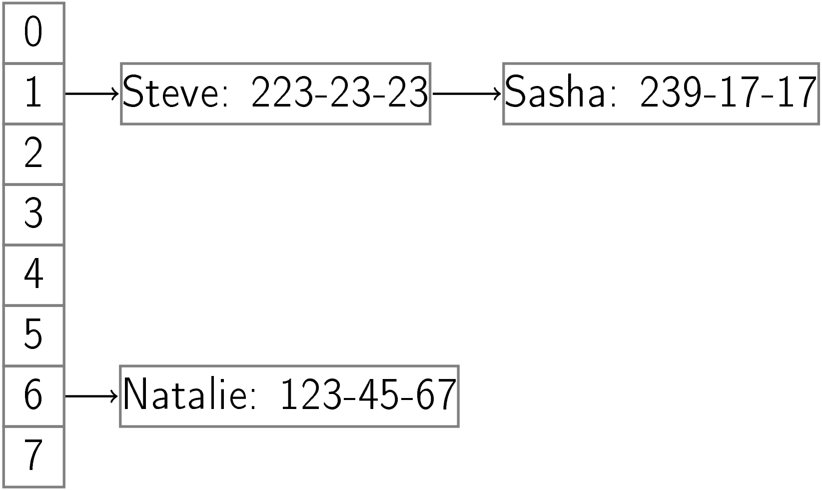

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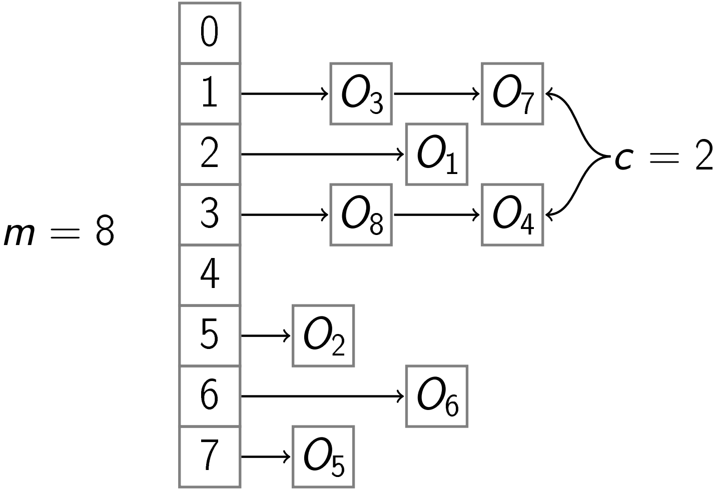
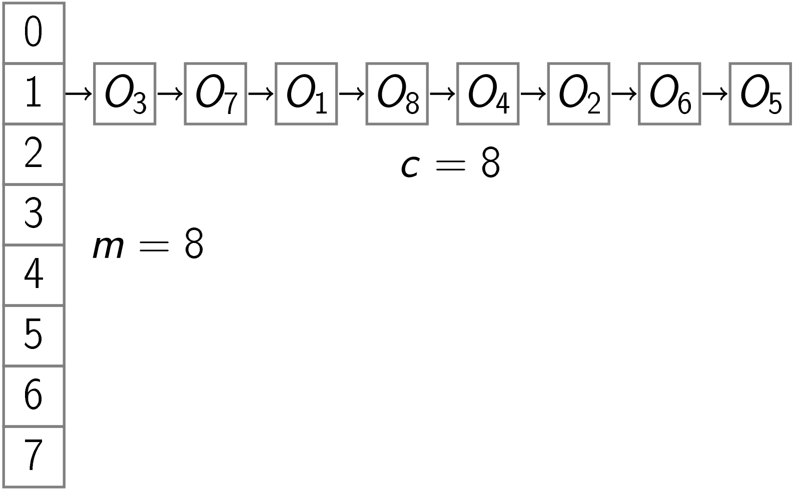
- You want small m and c!

Good Example



Bad Example



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- $h(800\text{-}123\text{-}45\text{-}67) = 800$
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- $h(425\text{-}234\text{-}55\text{-}67) =$
  $h(425\text{-}123\text{-}45\text{-}67) =$
  $h(425\text{-}223\text{-}23\text{-}23) = \cdots = 425$

Last Digits

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Last Digits

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- Problem if many phone numbers end with three zeros

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- Uniform distribution of hash values
- Different value when hash function called again — we won't be able to find anything!
- Hash function must be deterministic

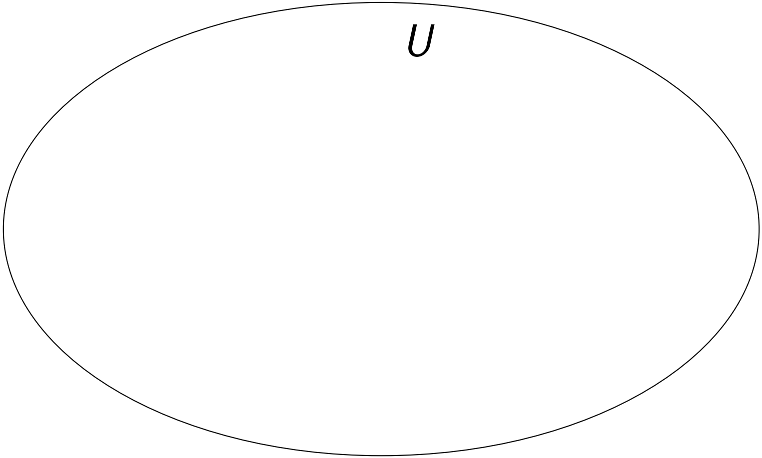
Good Hash Functions

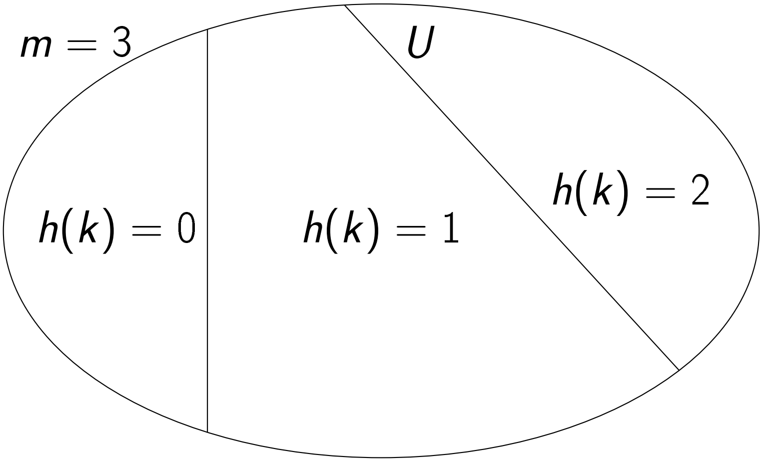
- Deterministic
- Fast to compute
- Distributes keys well into different cells
- Few collisions

No Universal Hash Function

Lemma

If number of possible keys is big ($|U| \gg m$), for any hash function h there is a bad input resulting in many collisions.


U



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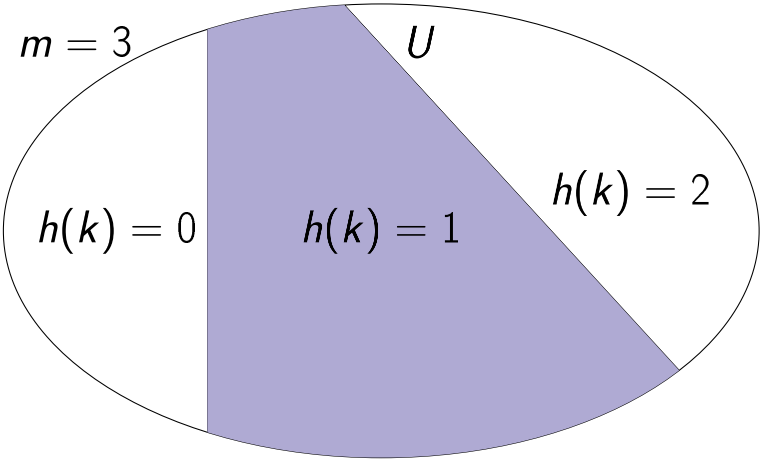
U
$h(k) = 0$
$h(k) = 1$
$h(k) = 2$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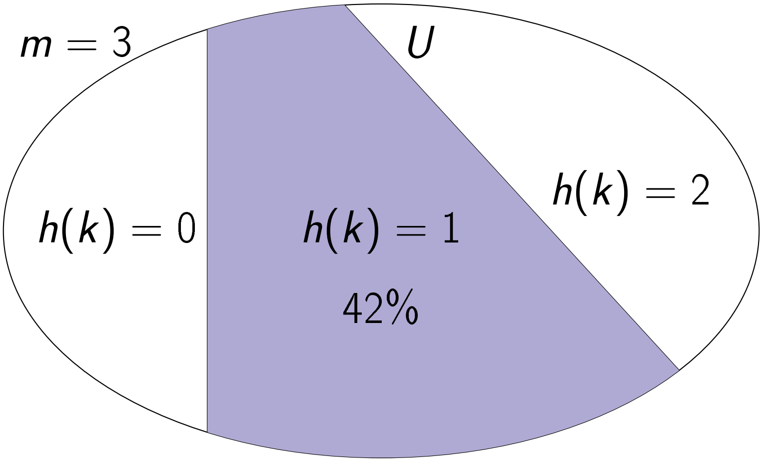
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is called a universal family if for any two keys $x, y \in U, x \neq y$ the probability of collision

$$Pr[h(x) = h(y)] \leq \frac{1}{m}$$

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means that a collision $h(x) = h(y)$ on selected keys x and y, $x \neq y$ happens for no more than $\frac{1}{m}$ of all hash functions $h \in \mathcal{H}$.

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- All hash functions in $\mathcal{H}$ are deterministic
- Select a random function h from $\mathcal{H}$
- Fixed h is used throughout the algorithm

Running Time

Lemma

If h is chosen randomly from a universal family, the average length of the longest chain c is $O(1+\alpha)$, where $\alpha = \frac{n}{m}$ is the load factor of the hash table.

Corollary

If h is from universal family, operations with hash table run on average in time $O(1+\alpha)$.

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- Resize the hash table when α becomes too large
- Choose new hash function and rehash all the objects

Keep load factor below 0.9:

Rehash(T)

$loadFactor \leftarrow \frac{T.\texttt{numberOfKeys}}{T.\texttt{size}}$

if $loadFactor > 0.9$:

  Create T_{new} of size $2 \times T$.size

  Choose h_{new} with cardinality T_{new}.size

  For each object O in T:

    Insert O in T_{new} using h_{new}

  $T \leftarrow T_{new}, h \leftarrow h_{new}$

Rehash Running Time

You should call `Rehash` after each operation with the hash table

Similarly to dynamic arrays, single rehashing takes $O(n)$ time, but amortized running time of each operation with hash table is still $O(1)$ on average, because rehashing will be rare

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  148-25-67 $\to$ 1 482 567
- Choose prime number bigger than 10^7, e.g. $p = 10\,000\,019$
- Choose hash table size, e.g. $m = 1\,000$

Hashing Integers

Lemma

$\mathcal{H}_p = \left\{ h_p^{a,b}(x) = ((ax + b) \bmod p) \bmod m \right\}$ for all $a, b : 1 \le a \le p - 1, 0 \le b \le p - 1$ is a universal family

Hashing Phone Numbers

Example

Select $a = 34$, $b = 2$, so $h = h_p^{34,2}$ and consider $x = 1\ 482\ 567$ corresponding to phone number 148-25-67. $p = 10\ 000\ 019$.

Hashing Phone Numbers

Example

Select $a = 34$, $b = 2$, so $h = h_p^{34,2}$ and consider $x = 1\,482\,567$ corresponding to phone number 148-25-67. $p = 10\,000\,019$.

$$(34 \times 1482567 + 2) \bmod 10000019 = 407185$$

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$407185 \bmod 1000 = 185$

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$407185 \bmod 1000 = 185$

$h(x) = 185$

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- Choose random hash function from universal family $\mathcal{H}_p$ (choose random $a \in [1, p-1]$ and $b \in [0, p-1]$)

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- Hash arbitrary strings of characters
- You will learn how string hashing is implemented in Java!

String Length Notation

Definition

Denote by $|S|$ the length of string S.

Examples

$|\texttt{``a''}| = 1$

$|\texttt{``ab''}| = 2$

$|\texttt{``abcde''}| = 5$

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- We should use all the characters in the hash function
- Otherwise there will be many collisions:
- For example, if $S[0]$ is not used, $h(\text{``aa''}) = h(\text{``ba''}) = \cdots = h(\text{``za''})$

Preparation

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Polynomial Hashing

Definition

Family of hash functions

$$\mathcal{P}_p = \left\{ h_p^x(S) = \sum_{i=0}^{|S|-1} S[i]x^i \bmod p \right\}$$

with a fixed prime p and all $1 \leq x \leq p-1$ is called polynomial.

PolyHash(S, p, x)

```
hash ← 0
for i from |S| − 1 down to 0:
    hash ← (hash × x + S[i]) mod p
return hash
```

Example: $|S| = 3$

1. $\texttt{hash} = 0$
2. $\texttt{hash} = S[2] \bmod p$
3. $\texttt{hash} = S[1] + S[2]x \bmod p$
4. $\texttt{hash} = S[0] + S[1]x + S[2]x^2 \bmod p$

Java Implementation

The method `hashCode` of the built-in Java class `String` is very similar to our `PolyHash`, it just uses $x = 31$ and for technical reasons avoids the $\pmod p$ operator.

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You now know how a function that is used trillions of times a day in many thousands of programs is implemented!

Lemma

For any two different strings s_1 and s_2 of length at most $L + 1$, if you choose h from $\mathcal{P}_p$ at random (by selecting a random $x \in [1, p - 1]$), the probability of collision $Pr[h(s_1) = h(s_2)]$ is at most $\frac{L}{p}$.

Proof idea

This follows from the fact that the equation $a_0 + a_1 x + a_2 x^2 + \cdots + a_L x^L = 0 \pmod{p}$ for prime p has at most L different solutions x.

Cardinality Fix

For use in a hash table of size m, we need a hash function of cardinality m.

First apply random h from $\mathcal{P}_p$ and then hash the resulting value again using integer hashing. Denote the resulting function by h_m.

Lemma

For any two different strings s_1 and s_2 of length at most $L+1$ and cardinality m, the probability of collision $Pr[h_m(s_1) = h_m(s_2)]$ is at most $\frac{1}{m} + \frac{L}{p}$.

Polynomial Hashing

Corollary

If $p > mL$, for any two different strings s_1 and s_2 of length at most $L + 1$ the probability of collision $Pr[h_m(s_1) = h_m(s_2)]$ is $O(\frac{1}{m})$.

Proof

$$\frac{1}{m} + \frac{L}{p} < \frac{1}{m} + \frac{L}{mL} = \frac{1}{m} + \frac{1}{m} = \frac{2}{m} = O\left(\frac{1}{m}\right) \quad \square$$

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- Computing `PolyHash`(S) runs in time $O(|S|)$
- If lengths of the names in the phone book are bounded by constant L, computing $h(S)$ takes $O(L) = O(1)$ time

Conclusion

- You learned how to hash integers and strings
- Phone book can be implemented as two hash tables
- Mapping phone numbers to names and back
- Search and modification run on average in $O(1)$!